UNCLASSIFIED

AD NUMBER

AD248574

LIMITATION CHANGES

TO:

Approved for public release; distribution is unlimited.

FROM:

Distribution authorized to U.S. Gov't. agencies and their contractors;

Administrative/Operational Use; 14 JUN 1960. Other requests shall be referred to Air Research and Development Command, Air Force Command and Control Development Division, Laurence G. Hanscom Field, Bedford, MA.

AUTHORITY

AFCRL ltr dtd 3 Nov 1971

UNCLASSIFIED

AD 248 574

Reproduced by the

ARMED SERVICES TECHNICAL INFORMATION AGENCY
ARLINGTON HALL STATION
ARLINGTON 12, VIRGINIA



UNCLASSIFIED

NOTICE: When government or other drawings, specifications or other data are used for any purpose other than in connection with a definitely related government procurement operation, the U. S. Government thereby incurs no responsibility, nor any obligation whatsoever; and the fact that the Government may have formulated, furnished, or in any way supplied the said drawings, specifications, or other data is not to be regarded by implication or otherwise as in any manner licensing the holder or any other person or corporation, or conveying any rights or permission to manufacture, use or sell any patented invention that may in any way be related thereto.

2 1 600 AFCCDD TH 60 34

SPACE TRACK

16 600

SPACE VEHICLE EPHEMERIS AND
DIFFERENTIAL CORRECTION PROGRAM
UNIFIED THEORY

ASTRODYNAMICS DEPARTMENT,
AERONUTRONIC
A DIVISION OF FORD MOTOR COMPANY
FORD ROAD, NEWPORT BEACH, CAL.,
CONTRACT AF19(604)-5885

14 JUNE 1960

496L SPO

AIR FORCE COMMAND AND CONTROL DEVELOPMENT DIVISION
AIR RESEARCH AND DEVELOPMENT COMMAND
UNITED STATES AIR FORCE



SATALOGED SY

SPACE TRACK

SPACE VEHICLE EPHEMERIS AND DIFFERENTIAL CORRECTION PROGRAM UNIFIED THEORY

ASTRODYNAMICS DEPARTMENT
AERONUTRONIC
A DIVISION OF FORD MOTOR COMPANY
FORD ROAD, NEWPORT BEACH, CALIF.
CONTRACT AF19(604)-5885

14 JUNE 1960

496L SPO
AIR FORCE COMMAND AND CONTROL DEVELOPMENT DIVISION
AIR RESEARCH AND DEVELOPMENT COMMAND
UNITED STATES AIR FORCE
LAURENCE G. HANSCOM FIELD, BEDFORD, MASS.

Air Force Command and Control Development Division
496L SP0

SPACE VEHICLE EPHEMERIS
and
DIFFERENTIAL CORRECTION PROGRAM
UNIFIED THEORY

Astrodynamics Department
Electronic Systems Operation
Aeronutronic
A Division of
Ford Motor Company
Ford Road, Newport Beach, California

ABSTRACT

This report traces the development of a differential correction theory for lunar and space probes. To overcome the singularities inherent in differential processes with conventional two-body descriptions of the path, as the eccentricity passes through unity in either the parabolic or rectilinear sense, a unified formulation of the two-body equations was developed. The differential correction procedure is based upon an ephemeris, integrated in Encke form by special perturbations, and upon differential expressions derived from the unified two-body equations. The report includes experimentation with simulated range, range-rate, azimuth, altitude, right ascension and/or declination data to evaluate the performance of the resulting IEM 709 program.

TABLE OF CONTENTS

SECTION	TITLE	PAGE
2	Introduction	1
	Technical Approach	2
	Experimentation	31

SECTION 1
INTRODUCTION

Under an amendment to contract AF19(604)-5885,
Aeronutronic has undertaken the development of a differential correction program for geocentric orbits of medium to high eccentricity, including lunar probes. Orbits of interest include the parabolic, elliptic, hyperbolic, and rectilinear forms, and an orbit theory which was applicable at once to all of these forms, without the singularities normally associated with the transition between these, was sought. This report discloses the development of a unified orbit theory especially adapted to machine computation; included are the necessary differential relationships and details of the Encke perturbation method.*

^{*} The proposal to employ a unified formulation of the two-body equations, and the major portion of the development thereof, are due to Samuel Herrick. Others on the staff of Aeronutronic who participated in this development include Jeannine Arsenault, R. H. Gersten, G. Matlin, C. Tross, C. T. Van Sant, G. B. Westrom, and L. G. Walters.

SECTION 2

TECHNICAL APPROACH

The Keplerian or two-body orbit is the basis for the perturbation methods of orbit integration and for the differential expressions required for orbit correction. In the perturbation method, the Keplerian path is employed as a reference path, and only the non two-body accelerations (or "perturbations") need be integrated to define the motion. Differential expressions involve the determination of linear cause-and-effect relationships between observation and orbit, and require a description of the orbit free of singularities in the region of interest. In both the perturbation method and differential correction use of other than two-body (or nearly two-body) form, such as position and velocity, severely limits the computational efficiency.

A review of the two-body orbit description betrays two serious singularities in the region of interest. First is the transition from ellipse to parabola to hyperbola as the eccentricity passes through unity and the semi-major axis becomes infinite. This region is described by three different sets of formulae, none of which is valid in this entire region. The situation is similar to the engineer's description of damped harmonic motion, wherein three mathematical descriptions cover the undamped, critically damped, and overdamped cases. A more subtle singularity occurs when the eccentricity passes through unity but the semi-major axis remains finite. The two-body orbits are degenerate forms of the ellipse and hyperbola, but the path is confined to a rectilinear path and a description which involves the orientation of the orbit plane fails.

The most familiar description of two-body motion is in terms of the six elements:

- a semi-major axis
- e eccentricity
- T time of perigee passage
- i inclination
- Ω longitude of the ascending node
- ω argument of perigee

This set of elements embodies the following singularities:

- 1. For zero eccentricity (i.e., the circular orbit) ω is indeterminate, and the anomalies used to describe position in the orbit plane have an indeterminate origin.
- 2. For zero inclination (i.e., the equatorial orbit), the longitude of the node is indeterminate, and the argument of perigee has an indeterminate origin.
- 3. For rectilinear motion, the orientation of the orbit plane is undefined; consequently, both the longitude of the node and the inclination are indeterminate.
- 4. For nearly parabolic motion, the equations for ellipse and hyperbola are singular for unit eccentricity, whereas the parabolic equations are invalid for non-unit eccentricity.

For the latter two reasons a description of the two-body motion by these familiar elements must be abandoned, and an alternate set of parameters chosen for this study. Incidentally, Aeronutronic has developed descriptions which also overcome singularities of the first two types, results of this research are published elsewhere.*

^{*}See, for example, Aeronutronic publication U-880.

The development of the orbit theory is given in the following sections. These include the basis for parameter selection, the Encke ephemeris formulation in terms of these parameters, the differential expressions for orbit correction, and the geometric transformations required for the utilization of topocentric observation residuals.

2.1 SELECTION OF PARAMETERS

Singularities of the types noted above arise from the parameters employed to describe the two-body orbit. This section will introduce a set of six parameters which removes all singularities save that for zero eccentricity from the theory, thereby unifying the separate descriptions heretofore required in the vicinity of unity eccentricity.*

To provide continuity from the elliptic to the hyperbolic region, including the parabolic (e = 1, infinite a) and rectilinear (e = 1, finite a) cases, it is necessary to adopt a uniform concept of mean motion n, ordinarily defined by:

$$n_e = k' \sqrt{\mu}$$
 (a) elliptic case $n_p = k' \sqrt{\mu}$ parabolic case $n_h = k' \sqrt{\mu}$ (-a) hyperbolic case

In the following, the parabolic definition for mean daily motion will be <u>adopted</u> for all three cases, and the remaining two-body equations will be adjusted accordingly. From this procedure evolves a set of two-body equations identical for the elliptic, parabolic, rectilinear, and hyperbolic cases, with no singularities as the eccentricity passes through unity.

Parallel research undertaken by Dr. Samuel Herrick of Aeronutronic has lead to an alternate development in terms of closed form f and g series expressions which is valid for all eccentricities, including zero, and all inclinations. Further development of this completely unified theory is continuing; the development of both a variation-of-parameters theory and of differential expressions for orbit correction will be pursued.

Beginning with the parabolic definition of mean daily motion, designated n,* the mean anomaly is related to time by

$$\tilde{M} - \tilde{M}_0 = \tilde{n} (t-t_0)$$

where the usual elliptic definition of M is $a^{-3/2}$ $\stackrel{\sim}{M}$. Kepler's equation, in elliptic form, is written

$$M = E - e \sin E = e (E - \sin E) + (1-e) E$$

or

$$\frac{\sim}{M} = a^{3/2} M = e^{3/2} V + q^{3/2} X$$

where, by definition,

$$\frac{\sim}{X}$$
 $\frac{\wedge}{a}$ E

$$\widetilde{U} = \frac{\widetilde{X}^3}{3!} - \frac{\widetilde{X}^5}{a \cdot 5!} + \frac{\widetilde{X}^7}{a^2 \cdot 7!} - \dots$$

The function U is derived from the (E-sin E) series; for negative a (hyperbolic case) this series may be derived from (sinh F - F) series. For infinite a, (parabola), the M form for Kepler's equation reduces to the familiar Barker's equation for the parabola. Thus the single form is adequate for all conics.

In this unified development, the tilde (\sim) symbol will be used to denote quantities whose definitions differ by virtue of the unification process, from familiar quantities in the conventional two-body theory. This comment does not apply to the usage in Section 2.4, where the tilde symbol is used to distinguish between equatorial and horizon coordinates of the unit vector triad \underline{L} , \underline{A} and \underline{D} .

The position vector $\underline{\boldsymbol{r}}$ is readily related to the above quantities through

$$\underline{\mathbf{r}} = \mathbf{x}_{\omega} \underline{\mathbf{P}} + \mathbf{y}_{\omega} \underline{\mathbf{Q}}.$$

To avoid the indeterminate nature of Q for the rectilinear case, where the semi-latus rectum $p = a \ (1-e^2)$ is zero and the orbit degenerates to a single line along \underline{P} , a vector $\underline{Q} * \sqrt{p} \ \underline{Q}$ is introduced such that

$$\underline{\mathbf{r}} = \mathbf{x}_{\omega} \underline{\mathbf{p}} + \frac{\mathbf{y}_{\omega}}{\sqrt{\mathbf{p}}} \widetilde{\mathbf{Q}}.$$

The coefficients are

$$x_{(1)} = a (\cos E - e) = a (1-e) - a (1-\cos E)$$

$$= q - C$$
where $C = \frac{x^2}{2!} - \frac{x^4}{a \cdot 4!} + \frac{x^6}{a^2 \cdot 6!} - \dots$

$$\frac{y_{(i)}}{\sqrt{p}} = \sqrt{a} \sin E = X - \frac{1}{a} \tilde{U}$$

Note that the elliptic expressions above, expressed in terms of circular functions, become hyperbolic functions for negative a, typical of the formulae for the hyperbola. The velocity expression is

$$\dot{\underline{r}} = \dot{x}_{w} P + \frac{\dot{y}_{w}}{\sqrt{P}} \tilde{Q}$$

where

$$\dot{x}_{(1)} = -\frac{\sqrt{\mu a}}{r} \sin E = -\frac{\sqrt{\mu}}{r} (x - \frac{1}{a} 0)$$

$$\frac{\dot{x}_{D}}{\sqrt{p}} = \frac{\sqrt{\mu}}{r} \cos E = \frac{\sqrt{\mu}}{r} (1 - \frac{1}{a} \tilde{C})$$

The unified theory presented here lends itself to automatic machine calculation, where transcendental functions are routinely computed from their Taylor series representations; on the other hand, lack of tables for $\mathbb U$ and $\mathbb C$ in terms of $\mathbb X$ and a^{-1} will discourage hand computation.

The parameters which suggest themselves for representation of all orbits (save zero eccentricity; note the usage of \underline{P}) are

$$\tilde{M}_{o}$$
, a^{-1} , \underline{P} , $\underline{\hat{Q}}$.

The previous development indicates the procedure for translating from parameters to position and velocity. The inverse transformation from position and velocity to these parameters follows:

Given
$$\underline{r}$$
, $\underline{\dot{r}}$, and t , to obtain \widetilde{M}_0 , $\frac{1}{a}$, \underline{P} and \widetilde{Q} :

$$r^2 = \underline{r} \cdot \underline{r}$$

$$r\dot{r} = \underline{r} \cdot \dot{\underline{r}}$$

$$\dot{s}^2 = \dot{r} \cdot \dot{r}$$

Compute
$$\frac{1}{a}$$
 and q: $\frac{1}{a} = \frac{2}{r} - \frac{\dot{s}^2}{\mu}$ (vis - viva integral)
$$p = (r^2 \dot{s}^2 - r^2 \dot{r}^2) / \mu$$

$$e^2 = 1 - p/a$$

$$q = p/(1 + e)$$

$$\widetilde{C} = \frac{1}{e} (r - q)$$

$$\widetilde{S} = r\dot{r}/e\sqrt{\mu}$$
Compute \underline{P} and \widetilde{Q} :
$$\underline{P} = [\frac{1}{r} (1 - \frac{1}{a} \widetilde{C})] \underline{r} - [\frac{1}{\sqrt{\mu}} \widetilde{S}] \dot{\underline{r}}$$

$$\widetilde{Q} = [\frac{1}{r} \widetilde{S}] \underline{r} + [\sqrt{\frac{1}{\mu}} (q - \widetilde{C})] \dot{\underline{r}}$$
Compute \widetilde{X} and \widetilde{M} :
$$\widetilde{S} = \widetilde{X} - \frac{\widetilde{X}^3}{a \cdot 3!} + \frac{\widetilde{X}^5}{a^2 \cdot 5!} - \frac{\widetilde{X}^7}{a^3 \cdot 7!} + \dots = \widetilde{X}$$

$$\widetilde{U} = \frac{\widetilde{X}^3}{3!} - \frac{\widetilde{X}^5}{a \cdot 5!} + \frac{\widetilde{X}^7}{a^2 \cdot 7!} - \dots$$

$$\widetilde{M} = q \widetilde{X} + e \widetilde{U}$$
Compute \widetilde{M}_0 :
$$\widetilde{n} = k' \sqrt{\mu}$$

2.2 ENCKE EPHEMERIS PROGRAM

The numerical integration of the perturbative accelerations, (and of the total acceleration in Cowell's method), is designated as "special perturbations". Factors to be considered in an orbit theory based on special perturbations include: what is to be integrated numerically, the choice of reference orbit, and the choice of the independent variable. In addition, the type of numerical integration procedure, e.g., Adams-Bashforth second-difference, second-sum Runge-Kutta, etc., must also be considered.

 $\widetilde{M}_0 = \widetilde{M} - \widetilde{n} (t - t_0)$

The special perturbations approach is particularly well suited to space trajectories which depart only slightly from orbits for which an analytical solution exists, e.g., a lunar trajectory.

$$\tilde{v} = \frac{s^3}{3!} + \frac{3^2 \tilde{s}^5}{a \cdot 5!} + \frac{3^2 \cdot 5^2 \tilde{s}^7}{a^2 \cdot 7!} + \dots$$

and

$$\widetilde{M} = \widetilde{U} + q\widetilde{S}$$

^{*} Alternatively, series for \widetilde{U} directly in terms of \widetilde{S} may be employed, i.e.,

In connection with what is to be integrated, a variety of methods for special perturbations have been developed. In effect, however, all of these methods use some modification or combination of the basic concepts characterized by (1) the integration of the total acceleration, as in Cowell's method, (2) integration of the departures from a fixed reference orbit, as in Encke's method, and (3) integration of the parameters of a varying reference orbit.

The Encke ephemeris program has been shown to be highly efficient for lunar trajectories. The program has demonstrated its ability to integrate trajectories to the moon with 15 integration steps with less than five miles accumulated truncation error at impact. Similar calculation with Cowell's method required almost 200 integration steps.

In the Encke orbit determination method only the departures from a two-body reference orbit are integrated. The position at some epoch (t) on the reference orbit is given by \underline{r}_e . The deviations of the vehicle from the reference orbit can then be given as (see Fig.1):

$$\underline{\rho} = \underline{r} - \underline{r}_{e}. \tag{2.2.1}$$

Considering one component we have the derivatives:

$$\frac{d^2\xi}{d\tau^2} = \frac{d^2x}{d\tau^2} - \frac{d^2x}{d\tau^2}$$

where

$$\frac{d^{2}x}{d\tau^{2}} = \ddot{x}_{e} = -\frac{\mu_{x}}{r^{3}}$$

$$\frac{d^{2}x}{d\tau^{2}} = \ddot{x} + \ddot{x}$$

$$\begin{cases} \ddot{x} = -\frac{\mu_{x}}{r^{3}} \\ \ddot{x} = \text{perturbations} \end{cases}$$

Finally

$$\frac{d^2\xi}{dz^2} = \mu \mu^2 \left(\frac{x_e}{r_e^3} - \frac{x}{r^3} \right) + \mu^2 \hat{x} \qquad \begin{cases} \xi \to \eta, \xi \\ x \to y, z \end{cases}$$
 2.2.2

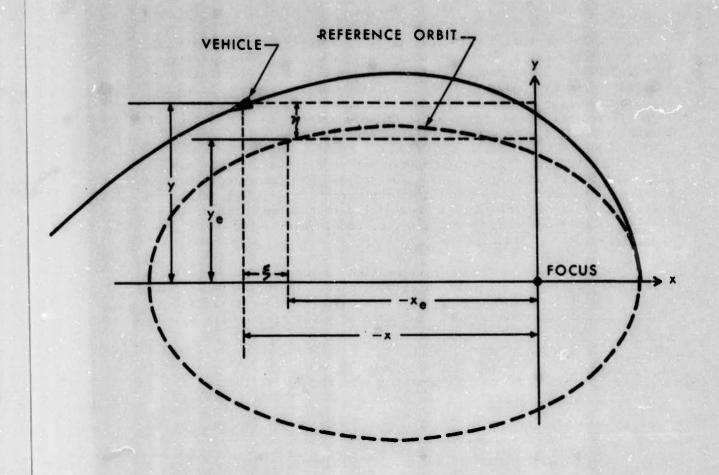


FIG. 1 DEVIATION FROM REFERENCE ORBIT

For small departures from the reference orbit, as in the case of lunar trajectories, the first term on the right side of the above equation is small and hence one integrates only slightly more than the \dot{x} term. The first term on the right side is normally developed into a series to avoid obtaining small differences of large quantities. This series is developed in the following manner:

First, $x = x_e + \xi$, etc., are substituted into the defining equation for r^2

$$r^2 = x^2 + y^2 + z^2$$

yielding

$$r^{2} = (x_{e} + \xi)^{2} + (y_{e} + \eta)^{2} + (z_{e} + \zeta)^{2},$$

$$= r_{e}^{2} + 2 \left[\xi(x_{e} + \frac{1}{2} \xi) + \eta \left(y_{e} + \frac{1}{2} \eta \right) + \zeta \left(z_{e} + \frac{1}{2} \zeta \right) \right]$$
2.2.3.

and q is taken to be

$$q = \frac{1}{r_e^2} \left[\xi \left(x_e + \frac{1}{2} \xi \right) + \eta \left(y_e + \frac{1}{2} \eta \right) + \zeta \left(z_e + \frac{1}{2} \zeta \right) \right], \qquad 2.2.4$$

so that equation 2.2.3 becomes

$$\left(\frac{r}{r}\right)^2 = 1 + 2 q$$
 2.2.5

The first term on the right-hand side of Equation 2.2.2 may be expressed as

$$\frac{x_{e}}{r_{e}^{3}} - \frac{x}{r^{3}} = \frac{1}{r_{e}^{3}} \left\{ \left[1 - \left(\frac{r_{e}}{r} \right)^{3} \right] \times - \xi \right\}$$
2.2.6

where, applying Equation 2.2.5,

$$\frac{r}{(\frac{e}{r})} = (1 + 2q)^{-3/2}$$
 2.2.7

Then, using a binomial expansion, Encke's series is defined by

$$1 - (\frac{r_e}{r})^3 = 1 - (1 + 2q)^{-3/2} = 3q - \frac{3 \cdot 5}{2!} \quad q^2 + \frac{3 \cdot 5 \cdot 7}{3!} \quad q^3 \quad \dots,$$

$$= fq \qquad 2 \cdot 2 \cdot 8$$

Finally, the substitution of Equations 2.2.6 and 2.2.8 into Equation 2.2.2 yields Encke's formula:

$$\frac{d^2 \xi}{d u^2} = \frac{\mu k^2}{r_e^3} \quad (fq x - \xi) + k^2 x^2, \quad \xi \rightarrow \eta, \zeta \qquad 2.2.9$$

The perturbative term x typically includes the second, third, and fourth harmonics of the earth's gravitational potential and the perturbations due to the sun and moon.

The procedure employed to calculate the position and velocity on the reference orbit from the unified parameters follows the pattern developed in Section 2.1.

2.3 DIFFERENTIAL EXPRESSIONS

This section deals with the development of linear differential expressions relating position and velocity to the unified orbit parameters introduced in Section 2.1. In Section 2.4, these relationships will be extended to include observation residuals referred to topocentric coordinates. In each of these operations, the algebraic manipulations are tedious, but straightforward. The object of this present section is vector relationships relating position and velocity increments $\Delta \mathbf{r}$ and $\Delta \mathbf{r}$ to increments in the parameters $\Delta \mathbf{r}$, $\Delta \mathbf{r}$ and $\Delta \mathbf{r}$. In order to remove redundancy, the latter two parameters will be developed in terms of related quantities which recognize the unit property of the vector \mathbf{r} and the orthogonality of \mathbf{r} and \mathbf{r} .

Differential expressions provide the link between observation residuals and corrections to the parameters. The development starts with the vectors of position and velocity:

$$\underline{\mathbf{r}} = \mathbf{x}_{\mathbf{W}} \underline{\mathbf{P}} + \frac{\mathbf{y}_{\mathbf{W}}}{\sqrt{\mathbf{p}}} \ \widetilde{\mathbf{Q}}$$
 2.3.1

$$\frac{\dot{\mathbf{r}}}{\dot{\mathbf{r}}} = \dot{\mathbf{x}} \frac{\mathbf{p}}{\omega} + \frac{\dot{\mathbf{w}}}{\sqrt{\mathbf{p}}} \quad \mathbf{Q}$$

where, for convenience, the coefficients of $\mathfrak Q$ will be denoted as $\mathfrak S$ and $\mathfrak S$, respectively. Differentiation leads to:

$$\Delta \underline{\mathbf{r}} = \underline{\mathbf{P}} \Delta \underline{\mathbf{x}}_{(1)} + \widehat{\mathbf{Q}} \Delta \widehat{\mathbf{S}} + \underline{\mathbf{x}}_{(1)} \Delta \underline{\mathbf{P}} + \widehat{\mathbf{S}} \Delta \widehat{\mathbf{Q}}$$
 2.3.3

$$\Delta \dot{\mathbf{r}} = \underline{\mathbf{P}} \Delta \dot{\mathbf{x}}_{\omega} + \underline{\mathbf{Q}} \Delta \dot{\mathbf{S}} + \dot{\mathbf{x}}_{\omega} \Delta \underline{\mathbf{P}} + \dot{\mathbf{S}} \Delta \underline{\mathbf{Q}}$$
 2.3.4

2.3.1 Differential Expressions for Position and Velocity Components in the Orbit Plane

In order to facilitate understanding of this differential procedure, those derivatives in equations 2.3.3 and 2.3.4 which pertain to the position and velocity components in the orbit plane, i.e., $\Delta x_{(1)}$, $\Delta \hat{S}$, $\Delta \dot{x}_{(1)}$, $\Delta \hat{S}$, will be derived first in terms of corrections to the parameters. Subsequently, the differentials associated with the expressions for the vectors defining the orientation of the orbit plane will be discussed. The \underline{P} and \underline{Q} vectors both define the orientation and serve as the chosen axes of the orbit plane.

The position and velocity components in the orbit plane, x_ω , y_ω and \dot{x}_ω , \dot{y}_ω , respectively, are defined.

$$\dot{x}_{\omega} = -\frac{\sqrt{\mu}}{r} \widetilde{S}$$

$$\dot{S} = \frac{\sqrt{\mu}}{r} (1 - \frac{1}{a} \widetilde{C})$$
2.3.6

The differential expressions for x_{ω} , \widetilde{s} , \dot{x}_{ω} and \widetilde{s} are obtained from the differentiation of equations 2.3.5 and 2.3.6:

$$\Delta \mathbf{x}_{(1)} = \Delta \mathbf{q} - \Delta_{\mathbf{C}}^{\infty}$$
 2.3.7

$$\Delta S = \Delta \widetilde{X} - \frac{1}{a} \Delta \widetilde{U} - \widetilde{U} \Delta (\frac{1}{a})$$
 2.3.8

$$\Delta \dot{\mathbf{x}}_{(1)} = \frac{\sqrt{\mu}}{\mathbf{r}} \left[\frac{\mathbf{S}}{\mathbf{r}} \Delta \mathbf{r} - \Delta \mathbf{S} \right]$$
 2.3.9

$$\Delta \dot{\hat{\mathbf{S}}} = -\frac{\sqrt{\mu}}{\mathbf{r}} \left[\frac{1}{\mathbf{a}} \Delta \tilde{\mathbf{c}} + \tilde{\mathbf{c}} \Delta \left(\frac{1}{\mathbf{a}} \right) + \frac{\dot{\hat{\mathbf{S}}}}{\sqrt{\mu}} \Delta \mathbf{r} \right]$$
 2.3.10

where the various terms are defined below. The determination of $\triangle C$ and $\triangle U$ follows from the series:

$$\widetilde{v} = \frac{\widetilde{x}^3}{3!} - \frac{\widetilde{x}^5}{a \cdot 5!} + \frac{\widetilde{x}^7}{a^2 \cdot 7!} - \dots$$

$$\widetilde{C} = \frac{\widetilde{X}^2}{2!} - \frac{\widetilde{X}^4}{a \cdot 4!} + \frac{\widetilde{X}^6}{a^2 \cdot 6!} - \dots$$

Differentiating, we obtain

$$\Delta \widetilde{\mathbf{U}} = \widetilde{\mathbf{C}} \quad \Delta \widehat{\mathbf{X}} - \widehat{\mathbf{U}}_{\mathbf{a}} \quad \Delta \left(\frac{1}{\mathbf{a}}\right)$$

$$\Delta \mathcal{C} = \mathcal{C} \quad \Delta \mathcal{X} - \mathcal{C}_{\mathbf{a}} \quad \Delta \left(\frac{1}{\mathbf{a}}\right)$$

where

$$\mathfrak{V}_{a} = \frac{\chi^{5}}{5!} - \frac{2}{a} \quad \frac{\chi^{7}}{7!} + \frac{3}{a^{2}} \quad \frac{\chi^{9}}{9!} - \dots$$

$$\mathcal{C}_{\mathbf{a}} = \frac{\chi^4}{4!} - \frac{2}{a} \frac{\chi^6}{6!} + \frac{3}{a^2} \frac{\chi^8}{8!} - \dots$$

From the formulae for e and r

$$e = 1 - \frac{1}{a} q$$

$$r = q + e \tilde{C}$$
,

differentiation yields

$$\triangle e = -q \triangle (\frac{1}{a}) - \frac{1}{a} \triangle q$$

and
$$\Delta r = \Delta q + e \Delta \tilde{c} + \tilde{c} \Delta e$$

or
$$\triangle \mathbf{r} = \mathbf{e} \, \hat{\mathbf{x}} \, \triangle \hat{\mathbf{x}} - (\mathbf{q} \hat{\mathbf{c}} + \mathbf{e} \hat{\mathbf{c}}_{\mathbf{a}}) \, \triangle (\frac{1}{\mathbf{a}}) + (1 - \frac{1}{\mathbf{a}} \, \hat{\mathbf{c}}) \triangle \mathbf{q}$$

Finally, the ΔX term is derived from the expressions for M:

$$\hat{M} = \hat{M}_0 + \hat{n} (t - t_0)$$

$$M = q \hat{X} + e \hat{V}$$

and their derivatives which follow

$$\triangle \hat{M} = \triangle \hat{M}_{o}$$

since $\triangle \hat{n} = 0$

$$\Delta \widetilde{M} = q \Delta \widetilde{X} + e \Delta \widetilde{U} + \widetilde{X} \Delta q + \widetilde{U} \Delta e$$

or $\triangle \hat{X} = \frac{1}{r} \triangle \hat{Y}_0 + \frac{1}{r} (q \hat{V} + e \hat{V}_a) \triangle (\frac{1}{a}) - \frac{1}{r} \hat{S} \triangle q$

At this point it is convenient to express Eq. 2.3.7-10 in terms of $\triangle \widetilde{M}_{o}$, $\triangle (\frac{1}{a})$ and $\triangle q$:

$$\Delta x_{\omega} = \widetilde{X}_{m} \Delta \widetilde{M}_{o} + \widetilde{X}_{aq} \Delta (\frac{1}{a}) + \widetilde{X}_{q} \Delta q \qquad 2.3.11$$

$$\Delta \tilde{S} = \tilde{S}_{m} \Delta \tilde{M}_{o} + \tilde{S}_{aq} \Delta (\frac{1}{a}) + \tilde{S}_{q} \Delta q \qquad 2.3.12$$

where

$$\widetilde{X}_{m} = -\widetilde{S}/r; \quad \widetilde{X}_{aq} = \widetilde{X}_{m} \quad (\widetilde{eU}_{a} + q\widetilde{U}) + \widetilde{C}_{a}; \quad \widetilde{X}_{q} = 1 - \widetilde{X}_{m}\widetilde{S}$$

$$\widetilde{S}_{m} = (1 - \frac{1}{a} \widetilde{C})/r$$
; $\widetilde{S}_{aq} = \widetilde{S}_{m} (e \widetilde{U}_{a} + q \widetilde{U}) - (\widetilde{U} - \frac{1}{a} \widetilde{U}_{a})$

$$\tilde{S}_{q} = -\tilde{S}_{m} \tilde{S}$$
 2.3.13

and
$$\Delta \dot{x}_{\omega} = \dot{x}_{m} \Delta \dot{y}_{0} + \dot{x}_{aq} \Delta (\frac{1}{a}) + \dot{x}_{q} \Delta q$$
 2.3.14

$$\Delta \hat{\mathbf{S}} = \hat{\mathbf{S}}_{m} \Delta \hat{\mathbf{M}}_{o} + \hat{\mathbf{S}}_{aq} \Delta (\frac{1}{a}) + \hat{\mathbf{S}}_{q} \Delta q \qquad 2.3.15$$

where $\dot{\vec{x}}_{m} = \frac{\sqrt{\mu} e^{\frac{\dot{x}^{2}}{r^{3}}} - \frac{\dot{\vec{S}}}{r}}{\dot{\vec{x}}_{q}} = \frac{\sqrt{\mu} e^{\frac{\dot{x}^{3}}{r^{3}}} + 2 \frac{\dot{\vec{S}} \dot{\vec{S}}}{r}}{\dot{\vec{x}}_{q}}$ $\dot{\vec{x}}_{aq} = \dot{\vec{x}}_{m} (e\hat{\vec{v}}_{a} + q\hat{\vec{v}}) - \frac{\sqrt{\mu} \dot{\vec{S}}}{r^{2}} (q\hat{\vec{c}} + e\hat{\vec{c}}_{a}) + \frac{\sqrt{\mu}}{r} (\hat{\vec{v}} - \frac{1}{a}\hat{\vec{v}}_{a})$ $\dot{\vec{S}}_{m} = -\frac{\dot{\vec{S}}}{r^{2}} (\sqrt{\frac{\mu}{a}} + e\hat{\vec{S}}) \qquad \dot{\vec{S}}_{q} = -\dot{\vec{S}} \dot{\vec{S}}_{m} - \frac{\dot{\vec{S}}^{2}}{\sqrt{\mu}}$ $\dot{\vec{S}}_{aq} = \dot{\vec{S}}_{m} (e\hat{\vec{v}}_{a} + q\hat{\vec{v}}) + \frac{\dot{\vec{S}}}{r} (q\hat{\vec{c}} + e\hat{\vec{c}}_{a}) + \frac{\sqrt{\mu}}{r} (\frac{1}{a}\hat{\vec{c}}_{a} - \hat{\vec{c}})$

To more directly relate the $\triangle \gamma_0$, $\frac{1}{2}$ $\triangle S$, etc., to the chosen parameters, the quantity $\triangle q$ is replaced by $\frac{1}{2}$ $\triangle p$. Note that:

$$\widetilde{Q} \cdot \widetilde{Q} = p$$

and

$$\widetilde{Q} \cdot \triangle \widetilde{Q} = \frac{1}{2} \triangle P$$

Then, from the expressions relating p, q, e and $\frac{1}{a}$

$$q = p/(1 + e)$$

$$e^2 = 1 - p/a$$

one obtains

$$\triangle q = \frac{1}{e} (\frac{1}{2} \triangle p) + \frac{q^2}{2e} \triangle (\frac{1}{a})$$

The substitution of the last expression will alter equations 2.3.11-16 in the following manner:

$$\Delta \underset{(1)}{\times} = \underset{(1)}{\tilde{\gamma}} \Delta \underset{(1)}{\tilde{\gamma}} + \underset{(2)}{\tilde{\gamma}} \Delta (\frac{1}{a}) + \underset{(1)}{\tilde{\gamma}} (\frac{1}{2} \Delta p) \qquad 2.3.17$$

$$\Delta \widetilde{S} = \widetilde{S}_{m} \Delta \widetilde{M}_{o} + \widetilde{S}_{ap} \Delta (\frac{1}{a}) + \widetilde{S}_{p} (\frac{1}{2} \Delta p) \qquad 2.3.18$$

where \widetilde{X}_m and \widetilde{S}_m remain as before, and

$$\widetilde{X}_{ap} = \widetilde{X}_{aq} + \frac{q^2}{2e} \widetilde{X}_{q} \qquad \widetilde{X}_{p} = \frac{1}{e} \widetilde{X}_{q}$$

$$\widetilde{S}_{ap} = \widetilde{S}_{aq} + \frac{q^2}{2e} \widetilde{S}_{q} \qquad \widetilde{S}_{p} = \frac{1}{e} \widetilde{S}_{q}$$
2.3.19

and

$$\Delta \dot{\mathbf{x}}_{(1)} = \dot{\hat{\mathbf{X}}}_{m} \Delta \hat{\mathbf{M}}_{o} + \dot{\hat{\mathbf{X}}}_{ap} \Delta (\frac{1}{a}) + \dot{\hat{\mathbf{X}}}_{p} (\frac{1}{2} \Delta p) \qquad 2.3.20$$

$$\Delta \dot{\hat{\mathbf{S}}} = \dot{\hat{\mathbf{S}}}_{m} \Delta \hat{\mathbf{M}}_{o} + \dot{\hat{\mathbf{S}}}_{ap} \Delta \left(\frac{1}{a}\right) + \dot{\hat{\mathbf{S}}}_{p} \left(\frac{1}{2}\Delta p\right)$$
 2.3.21

where again $\dot{\tilde{X}}_m$ and $\dot{\tilde{S}}_m$ remain unchanged and

$$\dot{\vec{x}}_{ap} = \dot{\vec{x}}_{aq} + \frac{q^2}{2e} \dot{\vec{x}}_{q} \qquad \dot{\vec{x}}_{p} = \frac{1}{e} \dot{\vec{x}}_{q}$$

$$\dot{\vec{s}}_{ap} = \dot{\vec{s}}_{aq} + \frac{q^2}{2e} \dot{\vec{s}}_{q} \qquad \dot{\vec{s}}_{p} = \frac{1}{e} \dot{\vec{s}}_{q}$$

$$2.3.22$$

Equations 2.3.17, 18, 20, 21 thus give expressions for Δx_{ω} , $\Delta \hat{S}$, $\Delta \dot{x}_{\omega}$, $\Delta \hat{S}$, which are different for each observation date, in terms of corrections to the elements, $\Delta \tilde{M}_{o}$, $\Delta (\frac{1}{a})$ and $\frac{1}{2}\Delta p$.

2.3.2 Differential Relationships Extended to the Orientation of the Orbit Plane

The orientation of the orbit plane is defined by \underline{P} , a unit vector directed toward perigee, and \underline{Q} , a vector normal to \underline{P} in a sense determined by the direction of motion. To avoid the indeterminate behavior of \underline{Q} for rectilinear motion, the vector

$$\tilde{Q} = \sqrt{p} Q$$

has been considered in this development.

The reduction of the six components of Δ <u>P</u> and Δ <u>Q</u> to fewer quantities leads to a definition of <u>P</u> in terms of its right ascension $\alpha*$ and declination $\delta*$. From these the unit vectors <u>A*</u> and <u>D*</u>, orthogonal to <u>P</u> (see Fig. 2) are calculated.

$$\underline{P} \begin{cases}
P_x = \cos \delta * \cos \alpha * \\
P_y = \cos \delta * \sin \alpha * \\
P_z = \sin \delta *
\end{cases}$$

$$\underline{\underline{A}}^* \begin{cases} A_x^* = -\sin \alpha^* \\ A_y^* = \cos \alpha^* \\ A_z^* = 0 \end{cases}$$

$$\underline{D}^* \begin{cases} D_x^* = -\sin \delta^* \cos \alpha^* \\ D_y^* = -\sin \delta^* \sin \alpha^* \\ D_z^* = \cos \delta^* \end{cases}$$

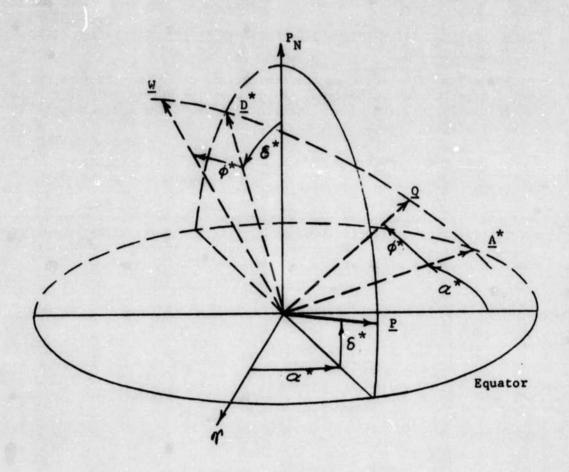


FIGURE 2

The differentiation of \underline{P} , $\underline{A}*$ and $\underline{D}*$ yields

$$\Delta \underline{P} = \underline{A} * \cos \delta * \Delta \alpha * + \underline{D} * \Delta \delta *$$

$$\Delta \underline{A} * = -\underline{P} \cos \delta * \Delta \alpha * + \underline{D} * \sin \delta * \Delta \alpha *$$

$$\Delta \underline{D} * = -\underline{A} * \sin \delta * \Delta \alpha * - \underline{P} \Delta \delta *$$
2.3.23

The angle $\emptyset *$, as shown in Fig. 2, allows us to derive the following relationships.

$$Q = \underline{A}^* \cos \emptyset^* + \underline{D}^* \sin \emptyset^*$$

$$\underline{W} = -\underline{A}^* \sin \emptyset^* + \underline{D}^* \cos \emptyset^*$$

$$\widetilde{Q} = \underline{A}^* c^* + \underline{D}^* s^*$$

$$2.3.24$$

2.3.25

where

$$c* = \sqrt{p} \cos \emptyset *$$
$$s* = \sqrt{p} \sin \emptyset *$$

The differential expression for & becomes

where
$$\underline{W}^* = c^* (\underline{D}^* \sin \delta^* - \underline{P} \cos \delta^*) - s^* \underline{A}^* \sin \delta^*$$

$$\underline{P}^* = -\underline{P} s^*$$

 $\Delta \mathbf{\hat{Q}} = \mathbf{\underline{A}} * \Delta \mathbf{c} * + \mathbf{\underline{D}} * \Delta \mathbf{s} * + \mathbf{\underline{W}} * \Delta \alpha * + \mathbf{\underline{P}} * \Delta \delta *$

A further reduction of unknowns in the expressions for $\Delta \times_{\omega}$, $\Delta \hat{\textbf{S}}$, $\Delta \dot{\textbf{x}}_{\omega}$ and $\Delta \hat{\textbf{S}}$ is carried out by noting that

$$\overset{\circ}{Q} \cdot \overset{\circ}{Q} = \frac{1}{2} \Delta p = c^* \Delta c^* + s^* \Delta s^*$$
2.3.26

Incorporating equation 2.3.26 into equations 2.3.17-22 and substituting these along with equations 2.3.23 and 25 into equations 2.3.3 and 4 gives the following formulae for $\Delta \underline{r}$ and $\Delta \underline{\dot{r}}$ in terms of six unknowns, namely, ΔM_0 , $\Delta (\frac{1}{a})$, $\Delta \alpha$, $\Delta \delta$, Δc and Δs .

$$\Delta \underline{r} = [\underline{P} \, \hat{X}_{m} + \underline{\hat{Q}} \, \hat{S}_{m}^{"}] \Delta \underline{\hat{M}}_{o} + [\underline{P} \, \hat{X}_{ap} + \underline{\hat{Q}} \, \hat{S}_{ap}^{"}] \Delta (\frac{1}{a}) \qquad 2.3.27$$

$$+ [c^{*} (\underline{P} \, \hat{X}_{p} + \underline{\hat{Q}} \, \hat{S}_{p}^{"}) + \underline{A}^{*} \, \hat{S}] \Delta c^{*}$$

$$+ [s^{*} (\underline{P} \, \hat{X}_{p} + \underline{\hat{Q}} \, \hat{S}_{p}^{"}) + \underline{D}^{*} \, \hat{S}] \Delta s^{*}$$

$$+ [\underline{A}^{*} \, x_{(i)} \, \cos \delta^{*} + \underline{W}^{*} \, \hat{S}] \Delta \alpha^{*} + [\underline{D}^{*} \, x_{(i)} + \underline{P}^{*} \, \hat{S}^{"}] \Delta \delta^{*}$$

and

$$\Delta \dot{\underline{r}} = \left[\underline{P} \dot{\hat{X}}_{m} + \dot{\underline{Q}} \dot{\hat{S}}_{m} \right] \Delta \hat{H}_{o} + \left[\underline{P} \dot{\hat{X}}_{ap} + \dot{\underline{Q}} \dot{\hat{S}}_{ap} \right] \Delta (\frac{1}{a}) \qquad 2.3.28$$

$$+ \left[c^{*} \left(\underline{P} \dot{\hat{X}}_{p} + \dot{\underline{Q}} \dot{\hat{S}}_{p} \right) + \underline{A}^{*} \dot{\hat{S}} \right] \Delta c^{*}$$

$$+ \left[s^{*} \left(\underline{P} \dot{\hat{X}}_{p} + \dot{\underline{Q}} \dot{\hat{S}}_{p} \right) + \underline{D}^{*} \dot{\hat{S}} \right] \Delta s^{*}$$

$$+ \left[\underline{A}^{*} \dot{\hat{S}}_{(1)} \cos \delta^{*} + \underline{W}^{*} \dot{\hat{S}} \right] \Delta \alpha^{*} + \left[\underline{D}^{*} \dot{\hat{S}}_{(1)} + \underline{P}^{*} \dot{\hat{S}} \right] \Delta \delta^{*}$$

The explicit derivation of α^* and δ^* is avoided by expressing the involved quantities \underline{A}^* and \underline{D}^* in terms of the given components of \underline{P} :

$$A_{z}^{*} = 0$$

$$D_{z}^{*} = (P_{x}^{2} + P_{y}^{2})^{1/2}$$

$$A_{x}^{*} = -P_{y}/D_{z}^{*}$$

$$A_{y}^{*} = P_{x}/D_{z}^{*}$$

$$A_y = P_x/D_z$$

$$D_x^* = -A_y^* P_z$$

$$D_y^* = A_x^* P_z$$

It is obvious from Fig. 2 and Eq. 2.3.24 that c and s may be obtained from the following dot products:

This completes the derivation of the differential expressions. In the following section, the topocentric coordinates of the observer are introduced, and scalar differential expressions between parameters and residuals in the observed quantities are derived. Details of the structure of the IBM-709 program are reserved to Sect. 3.

2.4 TOPOCENTRIC RESIDUALS

The position of the vehicle referred to the observer is related to the positions of the observer, R, and of the vehicle, r, both referred to the dynamical center by the fundamental expression.

 $\rho = r + R = \rho L$

2.4.1

where \underline{L} is a unit vector. (See Fig. 3)

The differential relationships relating residuals in the observed position to improvements of the orbital elements are embodied in the derivative of equation 2.4.1,

$$\Delta \rho = \Delta (\rho L) = \Delta \rho L + \rho \Delta L = \Delta r$$
. 2.4.2

It is assumed that there is no requirement for correcting the station vector, \underline{R} . The previous section developed an expression for $\underline{\wedge}\underline{\rho}$ in terms of corrections to the orbital parameters. In this section $\underline{\wedge}\underline{\rho}$ is translated into differential expressions involving residuals in the observations. In the discussion to follow, let the subscript c signify that the particular quantity is computed, that is, obtained from the representation, whereas no subscript denotes an observed quantity. The residuals will be taken in the sense "observed-minus-computed."

The observations considered in this study are ρ , A and h, α and δ , and $\dot{\rho}$. The differential correction program processes any combination and number of these observed quantities taken one at a time. This implies that although a complete set of observations may be available, for example, A, h and ρ , such that the components of $\Delta \rho$ could be obtained directly, the more general method employed in the differential correction program considers each piece of information without regard to the others. This procedure allows incomplete data to be processed and, furthermore, permits independent weighting or rejection of the data components.

First, if only one or two of the α , δ , and/or ρ are available it will be useful to define the auxiliary unit vectors \underline{A} and \underline{D} (Fig. 4) which form an orthogonal set with \underline{L} . \underline{D} is the north-pointing tangent to the celestial meridian at the point toward which \underline{L} is directed. \underline{A} is also tangent to the celestial sphere at the same point and parallel to the equator plane so that it completes the right-handed orthogonal set \underline{L} , \underline{A} , and \underline{D} . The components of \underline{L} , \underline{A} , and \underline{D} are

$$L_{x} = \cos \delta \cos \alpha$$

$$L_{y} = \cos \delta \sin \alpha$$

$$L_{z} = \sin \delta,$$

$$\underline{L}$$

$$2.4.3$$

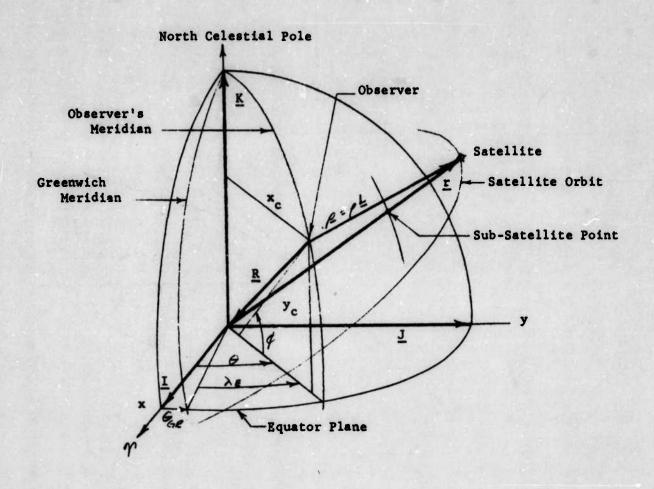


Figure 3

POSITION RELATIONSHIP OF OBSERVER, SATELLITE AND DYNAMICAL CENTER

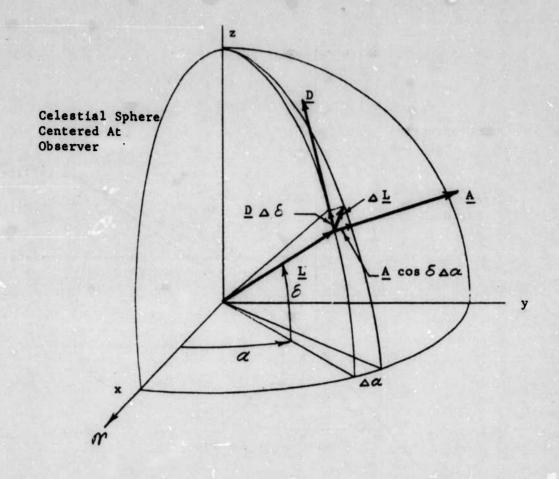


Figure 4
VECTOR RELATIONSHIPS

$$A_{x} = -\sin \alpha$$

$$A_{y} = \cos \alpha$$

$$A_{z} = 0,$$

$$\Delta$$
2.4.4

and

$$D_{x} = -\sin \delta \cos \alpha$$

$$D_{y} = -\sin \delta \sin \alpha$$

$$D_{z} = \cos \delta,$$

$$D_{z} = \cos \delta,$$

where α and δ are the topocentric right ascension and declination of the object. In addition, \underline{L}_c is determined from

$$\underline{L}_{c} = \underline{\rho}_{c} / \rho_{c}$$
 2.4.6

where

$$\underline{\rho}_{c} = \underline{r} + \underline{R}$$

and

$$\rho_{\rm c}^2 = \rho_{\rm c} \cdot \rho_{\rm c}$$

For angular observations, the residuals in the direction cosines, ΔL , are obtained from

$$\Delta \underline{L} = \underline{L} - \underline{L}_{c} .$$

Also it can be seen from Fig. 4 that

 $\Delta \underline{L} = \underline{A} \triangle \alpha \cos \delta + \underline{D} \triangle \delta$

2.4.7

Successive dot products of \triangle <u>L</u> with <u>A</u> and <u>D</u> will lead to scalar differential expressions for $\triangle \alpha \cos \delta$, and $\triangle \delta$, respectively.

 $\Delta \underline{L} \cdot \underline{A} = \Delta \alpha \cos \delta,$ 2.4.8

 $\Delta \underline{L} \cdot \underline{D} * \Delta S \qquad 2.4.9$

Similarly, the dot product of $\Delta\!\varrho$ with \underline{L}_c relates a slant-range residual $\Delta\!\varrho$ to improvements of the orbit parameters.

 $\Delta \underline{\mathbf{p}} \cdot \underline{\mathbf{L}}_{\mathbf{c}} = \Delta \mathbf{p}. \qquad 2.4.10$

Differential expressions involving observations in a horizon-oriented coordinate system may be obtained in a similar manner by defining the horizon system unit vector triad \underline{L} , $\underline{\widetilde{A}}$, and $\underline{\widetilde{D}}$, whose components are:

$$L_{xh} = -\cos A \cos h$$

$$L_{yh} = \sin A \cos h$$

$$L_{zh} = \sin h,$$

$$2.4.11$$

$$\widetilde{A}_{xh} = \sin A$$

$$\widetilde{A}_{yh} = \cos A$$

$$\widetilde{A}_{zh} = 0,$$
2.4.12

and

where h is the altitude or elevation angle and A is the azimuth, measured in the positive sense to the east from north. These horizon system components are rotated into the equatorial system, to which the components of $\Delta \rho$ are referred by

$$L_{x} = L_{xh} S_{x} + L_{yh} E_{x} + L_{zh} Z_{x}$$

$$\widetilde{A}_{x} = \widetilde{A}_{xh} S_{x} + \widetilde{A}_{yh} E_{x} - \widetilde{A}_{zh} Z_{x}$$

$$\widetilde{D}_{x} = \widetilde{D}_{xh} S_{x} + \widetilde{D}_{yh} E_{x} + \widetilde{D}_{zh} Z_{x}$$

$$(x \longrightarrow y, z.) \qquad 2.4.14$$

The components of the auxiliary unit vectors \underline{S} , \underline{E} and \underline{Z} , directed to the south, east, and zenith, respectively, are

$$S_{x} = \sin \phi \cos \theta$$

$$S_{y} = \sin \phi \sin \theta$$

$$S_{z} = -\cos \phi$$

$$E_{x} = -\sin \theta$$

$$E_{y} = \cos \theta$$

$$E_{z} = 0$$

$$E_{z} = 0$$

and

$$Z_{x} = \cos \phi \cos \theta$$

$$Z_{y} = \cos \phi \sin \theta$$

$$Z_{z} = \sin \phi$$

$$Z_{z} = \sin \phi$$

where $\dot{\varphi}$ is the observer's astronomical latitude and θ is the local hour angle of the vernal equinox. Then

$$\Delta \underline{L} = \underline{\hat{A}} \triangle A \cos h + \underline{\hat{D}} \triangle h.$$
 2.4.18

Scalar differential expressions for \triangle A cos h, and \triangle h can now be obtained by forming the dot products of $\triangle L$ with \underline{A} and \underline{D} , respectively:

$$\Delta L \cdot \frac{\Delta}{A} = \Delta A \cos h$$
 2.4.19

and

$$\Delta \underline{L} \cdot \underline{\hat{D}} = \Delta h.$$
 2.4.20

Thus, residuals in the observations of angular position in the horizon coordinate system can be related to corrections to the orbit parameters through $\Delta \rho$.

The corrections to the orbit parameters are introduced through $\Delta \underline{\hat{\rho}}$ and $\Delta \underline{\hat{\rho}}$ as developed in Sect. 2.3.

SECTION 3 EXPERIMENTATION

The unified ephemeris and differential correction theory has been subjected to experimental verification to determine the cacuracy and rate of convergence with various types of simulatei observations.

EPHEMERIS PROGRAM

The numerical integration program was checked against the Encke Moonshot program. The major differences between the two methods are the choice of parameters used to define the two-body reference orbit and the argument used for the numerical integration, i.e., time for the unified Encke program and eccentric anomaly, E, for the Encke Moonshot program. The results of the comparison are listed in Table 1.

Further experimentation is being conducted to determine integration step-size, time for integration, etc.

DIFFERENTIAL CORRECTION PROGRAM

The program has been verified by correcting a hyperbolic orbit to one that is elliptical with various combinations of range and angle data. Tables 2 through 4 give the sizes of the r.m.s. residuals and list the values of the parameters M, $\frac{1}{a}$, P and Q after each of four corrections. The results of Table 2 were based on 32 range observations spaced over 7 hours and 18 minutes. Over the same time-span, Table 3 involves 15 observations of Q and Q are reproductions of the program listing of the residuals in the range and angular observations after one correction.

Further verification of the program was carried out by correcting from an elliptic to a parabolic orbit and from a hyperbolic to a parabolic orbit. The success of this experimentation, orbit correction through unit eccentricity, satisfies the definition of the unified theory. Tables 6, 7 and 8 list the values of the parameters for the elliptic case and the r.m.s. residuals in the observations after each correction for both the elliptic and hyperbolic runs. In all cases the observations were obtained over a 6 hour and 42 min. period. Thirty-two range observations were used in Table 6, 13 sets of α and δ and 15 sets of A and h for Table 7, and a combination of 32 range, 13 α and δ and 14 A and h observations are included in Table 8. Tables 9 through 12 demonstrate the use of the program with range rate simulated observations.

TABLE 1

COMPARISON OF ENCKE MOONSHOT PROGRAM TO UNIFIED ENCKE INTEGRATION PROGRAM

Position (Earth Radii)

	z component		25.385243 25.385280		27,504982 27,503995
LOSICION (Parcii want)	y-component		2.828210 2.8281408		-1,6678242 -1,6668810.
	x-component	At = 54.5 min.	Moonshot -15.590354 Unified -15.590443	$\Delta t = 6 \text{ hr. } 38/\text{min.}$	Moonshot -19.706187 Unified -19.705873

Velocity (Earth Radii/kelmin.)

z component		.093933927
y-component		17530767
x-component	Δt = 54.5 min.	Moonshot16817206 Unified16816788

	.072969333 .072900019
	1790717
$\triangle t = 6 \text{ hr. } 38.1 \text{ min.}$	Moonshot15423712 Unified -15421567

 Δt implies the interval of time since start of integration.

TABLE 2

RANGE OBSERVATIONS

					₹ 0°	4.7159303	4.669013	4.6690086	
R.M.S. Residuals (n. mi.) 1322.0093 5.0605621 .0039910565	.0020732041	1. a	.0010127173	.00099998600 .0010001812	50 ⁵ ⁄	-1.382 4660	-1.368 7039	-1.368 6811	
R.M.S. Re 1322. 5.	PARAMETERS		.00.	.000	₹	-4.0309788	-3.9908506	-3.9908520	
Iterations 1 2 3	4	*Z°0	98.676566 98.678646 98.676728	98.676532 98.676890	Q,	.44120929	.44120464	.44120588	
It			8 8 8	8 8 8	P _Y	.87062552	.87062869	.87062640	
					ě,	.21759032	.21758707	.21758879	

TABLE 3

ANGULAR OBSERVATIONS

					5 0 %	4.7159303 4.6689058 4.6690010 4.6690018 4.6690015
als (n. mi.)	.43512 .0306040 .0057359097 .0059121116			99998 1525 2501 1545 1564	%	-1.3824660 -1.3691051 -1.3687224 -1.3687238
R.M.S. Residuals (n. mi.	104,43512 8,0306040 00573590	ERS	⊣ 1 ø	00099999998 .0010091525 .0010002501 .0010001545	⊹ ^×	-4.0309788 -3.9910485 -3.9908734 -3.9908747
SI.		PARAMETERS			e N	.44120929 .44126089 .44121044 .44121056
Iterations	T 7 R 4		²z°	98.676566 98.673321 98.676874 98.676775	۳۶	.87062552 .87060935 .87062513 .87062510
					٣×	.21759032 .21755033 .21758945 .21758930

TABLE 4

RANGE AND ANGULAR OBSERVATIONS

~					ζο ^N	4.7159303	4.6690044	
R.M.S. Residuals (n. mi.) 768.01078 7.2070230	.0050729365	⊢ ø	0009999998 .0010100009 .0010000813	.0010000401	58 _y	-1.3824660	-1.3687064 -1.3687094 -1.3687101	
R.M.S. Res 768.0	PARAMETERS		00	.000	⁵ o _x	-4.0309788	-3.9908663 -3.9908678 -3.9908683	
Iterations 1	3 4 <u>PAR</u>	γ ≖ °	98.676566 98.674281 98.676704	98.676647	er N	.44120929	.44120827 .44120861 .44120871	
비			86 6 8 8 8 8 8 8 8 8 8 8 8 8 8 8 8 8 8	98.	P	.87062552	.87062581 .87062571 .87062566	
					۳×	.21759032	.21759121 .21759087 .21759081	

ELEV RESID	10292940+02 91413201+01 9240679+01 66765469+00 -21983465+01 -47009890+01 -47009890+01 -25289572+01 -25289572+01 -25289572+01 -264628+01 -2618736+01 -26199713+01 -14199713+01 -52918736+01 -69224628+01 -1486675+02
AZIM RESID	.53597231+01 .71339336+01 .11003844+02 .11335681+02 .1043355+02 .1043355+02 .10423414+02 .11356898+02 .11425414+02 .11356898+02 .11425414+02 .1142415+01 .10203741+02 .11491429+02 .1149144+01 .114897377+01
1 DECL RESID	36172242+01 49888399401 51829651+01 53581421+01 36138554+01 50499811+01 50499811+01 50499811+01 50499811+01 519854+01 519854+01 519854+01 519854+01 41485925+01 47269503+01
PAGE 001 R.A. RESID	11019095+02 1028058235+02 10097273+02 11083844+02 1108838+02 11325433+02 11325433+02 1132543402 1132543402 1132543402 1132543402 1132543402 10335919+02
ON ENCKE EPHEMERIS OBSERVATIONS RODOT RESID	
	.29970559+01 .50334116+01 .54866550+01 .598264695+01 .30627448+01 .30627448+01 .26981774+01 .26981774+01 .269877651+01 .44627393+01 .59924695+01 .59334116+01 .59924695+01 .599246961 .599246961 .599246961 .599246961 .599246961 .599246961 .599246961 .599246961 .599246961 .599246961 .599246961 .599246961 .599246961 .599396915+01 .599396915+01 .599396915+01 .599396915+01
	• 14400000+04 • 16420000+04 • 16420000+04 • 17460000+04 • 17700000+04 • 17700000+04 • 18420000+04 • 18420000+04 • 18480000+04 • 18480000+04
UNIFIED DIFFERENTIAL GENERALIZED INPUT MAY	тавье 37

TABLE 6
RANGE OBSERVATIONS

(n. mi.) Hyperbolic to Parabolic	655.35633 1.1399132 .0017279604 .0013167071		<u>1</u>	.00099999998 .00000783011 .00000015573 00000001529	2 00 €	4.6926192 4.6925269 4.6925320 4.6925322
Residuals		E)		00000.	,°∿	-1.3687096 -1.3752158 -1.375843 -1.3756190 -1.3756035
R.M.S. Elliptic to Parabolic	658.93944 1.4511371 .0037209759	PARAMETERS (ELLIPTIC CASE)			∂o×	-3.9908683 -4.0106121 -4.0109648 -4.0109665
E11		PARAMETE			PZ	.44120929 .44112300 .44120727 .44120668
ations	432		⊱ °	98.676566 98.683683 98.676826 98.676532	ъ _р	.87062552 .87066951 .87062573 .87062817
<u>Iterati</u>				98.676 98.683 98.676 98.676	^d ×	.21759032 .21758923 .21759353 .21758494 .21758728

TABLE 7

ANGULAR OBSERVATIONS

R.M.S. Residuals (n. mi.) Elliptic to Parabolic Hyperbolic to Parabolic	50.045619 1.7554224 0.0044739828 0.005939042 1.0050587135	PARAMETERS (ELLIPTIC CASE)	a ! 1	.0009999998 .00000555180 .00000005136 .00000009425	$\begin{array}{cccccccccccccccccccccccccccccccccccc$
Iterations	7 2 5 7		₹°	98.676566 98.677202 98.676638 98.676657	P, Py .21759032 .87062552 .21754839 .87061224 .21758920 .87062513 .21758891 .87062495

RANGE AND ANGULAR OBSERVATIONS

R.M.S. Residuals (n. mi.) Elliptic to Parabolic Hyperbolic to Parabol	381.98980 1.7044480 .0040398233 .0040581253 .0049367466	PARAMETERS (ELLIPTIC CASE)	1 E	.0009999998 .00000312037 .00000001234 00000001002	% % % %	-3.9908683 -1.3687096 4.6690040 -4.0111155 -1.3759465 4.6924468 -4.0109724 -1.3756038 4.6925258 -4.0109734 -1.3756059 4.6925257 -4.0109733 -1.376058 4.6925257
		PAF			D ₁	.44120929 .44125102 .44120933 .44120954
Iterations	t 3 5 1		₹ <mark>Σ</mark> °	98.676566 98.675255 98.676569 98.676539 98.676542	a, ^S	.87062552 .87061389 .87062551 .87062545
				88888	ďχ	.21759032 .21755218 .21759023 .21758997 .21759000
				40		

TABLE 9

Residuals (Elliptic to Parabolic)

Time	1st Iteration	2nd Iteration	3rd Iteration
152% 0000	60 862755	.36555847	00014492869
1524 0000	60.862755	.36555847	00014492869
1440 0000	57,372751	. 28476072	000024154782
1440.0000	57.372751	.28476072	000024154782
1488,0000	57.576376	.28125828	00019323826
1488,0000	57.576376	.28125828	00019323826
1542,0000	57.953650	. 28067856	.000048309564
1596,0000	58.468871	. 28299742	000048309564
1596,0000	58.468871	. 28299742	000048309564
1644.0000	59.015398	. 28734528	000048309564
1644.0000	59.015398	. 28734528	- 000048309267
1692,0000	59.611586	.29372215	
1692,0000	59.611586	. 29372215	
1692,0000	59.611586	. 29372215	
1746.0000	60.295649	.30294927	.00014492869
1746.0000	60.295649	.30294927	.00014492869
1794.0000	60.872369	.31207978	.00014492869
1842,0000	61.382663	.32198324	.00014492869
1842.0000	61.382663	.32198324	.00014492869
1470,0000	57.549492	. 26893934	.000048309562
1470.0000	57.549492	. 26893934	.00004830956
1,560,0000	58.792545	. 28309404	
1560,0000	58.792545	. 28309404	
1770.0000	61.311309	.33125868	.00019323826
1770.0000	61.311309	.33125868	.00019323826
1632.0000	60.839325	.32903644	00014492869
1632.0000	60.839325	.32903644	00014492869
1758,0000	61.510538	.35575163	
1758.0000	61.510538	, 35575163	

TABLE 10

Parameters (Elliptic to Parabolic)

		}o⁵°	4.6990040 4.6924119 4.6925275 4.6925273
	8 185 183483 810156	φ,	-1.3687096 -1.3752721 -1.3756097 -1.3756078
⊢ ¤	1 .0009999998 .0000073683185 .0000000033183483		-3.9908683 -4.0106505 -4.0109718 -4.0109738
		e Z	.44120929 .44116342 .44120838
} _E °	98.676566 98.673802 98.676632 98.676655	Py	.87062552 .87064438 .87062639
		ď _X	. 21759032 . 21760784 . 21758860 . 21758965

TABLE, 11

Residuals (Hyperbolic to Parabolic)

Tine	lst Iteration	2nd Iteration	3rd Iteration
159% 0000	-60.385215	.35188686	000096619128
1524 0000	-60,385215	.35188686	000096619128
16.60 0000	- 56.951081	. 28048532	
0000	- 56.951081	. 28048532	0
1,88 0000	- 57.145865	.27425339	00014492869
1,88 0000	- 57.145865	. 27425339	00014492869
154.2 0000	- 57.510506	. 26966398	.000096619128
1596 0000	- 58.011621	. 26739343	000096619128
1596,0000	- 58.011621	. 26739343	000096619128
1644,0000	- 58.544427	. 26753836	- ,000048309364
1644,0000	- 58.544427	. 26753836	- ,000048309384
1692,0000	- 59.126074	. 26956736	000048309364
1692, 2000	- 59,126074	. 269 56736	000048309564
1692,0000	- 59.126074	. 26956736	000048309364
1746.0000	- 59.793616	. 27396353	.00014492869
1746.0000	- 59.793616	. 27396353	.00014492869
1794,0000	-60.357533	. 27898773	. 000048309564
1842 0000	-60.856475	. 28502642	.00014492869
1842 0000	-60.856475	.28502642	.00014492869
1470.0000	- 57, 116590	. 26386683	000048309564
1470.0000	- 57.116590	. 26386683	000048309564
1560.0000	- 58.330223	.27000215	.000096619128
1560.0000	- 58.330223	. 27000215	.000096619128
1770.0000	-60.793528	.30067872	.00014492869
1770.0000	-60.793528	.30067872	.00014492869
1632 0000	-60.341156	.30937444	- 000048309564
1632 0000	-60.341156	.30937444	-,000048309564
1758.0000	-60.998408	.32811856	-,000096619128
1758,0000	-60.998408	.32811856	-,000096619128

TABLE 12

Parameters (Hyperbolic to Parabolic)

		}⊘ ^N	4.7159303 4.6924738 4.6925275 4.6925286
l æ	0009999998 .0000047848298 00000020548157 000000031070959	}o [≯]	-1.3824660 -1.3755088 -1.3756110 -1.3756084
			-4.0309788 -4.0108532 -4.0109720 -4.0109743
		en N	.44120929 .44119870 .44120871 .44120910
₹°	98.676566 98.675465 98.676582 98.676670	ф У	.87062552 .87062886 .87062633
		М×	.21759032 .21759838 .21758821 .21758938